CHAPTER - 4

STEADY ELECTRIC CURRENTS

— Chapter-4 "Steady Electric Currents" —

Exercise 4.1 $R_{al} = 0.0625$ cm $\quad R_{cu} = 0.125$ cm (radius)

$$|\vec{J}| = \frac{I}{A} \Rightarrow \quad J_{al} = \frac{8\times10^{-3}}{\pi \times 0.0625^2 \times 10^{-4}} = 6.52 \ kA/m^2$$

$$J_{cu} = 1.63 \ kA/m^2$$

Exercise 4.2 $L = 100$ km. $R_{cu} = 1.5$ cm, $I = 1000A \Rightarrow J = \frac{1000}{\pi R_{cu}^2} = 1.415 \ MA/m^2$

$$J = \sigma_{cu} E \Rightarrow E = \rho_{cu} J = 1.72\times10^{-8} \times 1.415\times10^{6} = 24.34 \text{ mV/m}$$

$$u_e = \frac{e\tau}{m} = \frac{1.6\times10^{-19} \times 2.7\times10^{-14}}{9.1\times10^{-31}} = 4.75\times10^{-3}$$

Hence, $u = u_e E = 115.61 \times 10^{-6}$ m/s $\Rightarrow t = \frac{100\times10^{3}}{u} = 865\times10^{6}$ s or 27.4 years

Exercise 4.3 $u = 1.5\times10^{6}$ m/s, $J = 5 A/mm^2 = 5\times10^{6} A/m^2$

since $\vec{J} = \rho\vec{u}$, $\rho = \frac{J}{u} = 3.333 \ C/m^3$

# of electrons: $N = \frac{3.333}{1.6\times10^{-19}} = 20.83\times10^{18}$ electrons/m^3

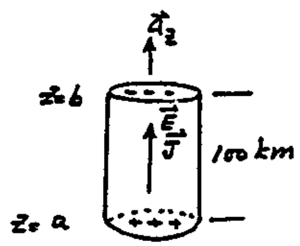
Exercise 4.4 $r = 2$ cm, $I = 100A \Rightarrow J_z = \frac{I}{\pi r^2} = 79.577\times10^{3} \ A/m^2$

$$E_z = \rho_{al} J_z = 79.577\times10^{3} \times 2.83\times10^{-8} = 2.252 \text{ mV/m}$$

$$V_{ab} = -\int_b^a \vec{E}\cdot\vec{dl} = -\int_b^a E_z dz = E_z(b-a)$$

$$= 2.252\times10^{-3} \times 100\times10^{3} = 225.2 \text{ V}$$

$$R = \frac{V}{I} = 2.25 \ \Omega$$



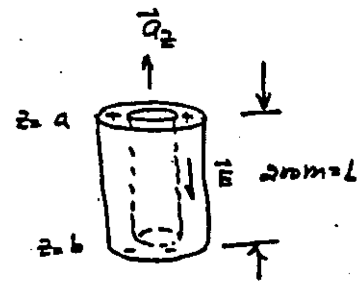
Exercise 4.5 $E = -10\times10^{-3}\vec{a}_z$ V/m, $A = \pi(s_2^2 - s_1^2) = 21\pi \text{ cm}^2$

$$V_{ab} = -\int_b^a \vec{E}\cdot dl = 10\times10^{-3} \times 200 = 2V$$

$$\vec{J} = \sigma\vec{E} = \frac{\vec{E}}{\rho} = -\frac{10\times10^{-3}}{8.9\times10^{-8}}\vec{a}_z = -112.36\,\vec{a}_z \ kA/m^2$$

$$I = \int_s \vec{J}\cdot ds = 112.36\times10^{3} \times 21\pi\times10^{-4} = 741.28 A \qquad \text{Also } R = \frac{\rho L}{A} = 2.7 \text{ m}\Omega$$

$$R = V_{ab}/I = 2.7 \text{ m}\Omega$$



Exercise 4.6 $\quad A_{cu} = \frac{\rho L}{R} = \frac{1.72\times10^{-8}\times 200}{2.7\times10^{-3}} = 1.274\times10^{-3}\ m^2$

If a is the radius of Copper wire: $a = \sqrt{\frac{1.26\times10^{-3}}{\pi}} = 20.14$ mm

For $V = 2V$, $E = 10\ mV/m$ and $J = \sigma E = \frac{E}{\rho} = 581.4\ kA/m^2$

Exercise 4.7 $\quad \vec{J} = \frac{kV_o}{M\rho}\vec{a}_\rho \Rightarrow \nabla\cdot\vec{J} = \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\left(\frac{kV_o}{M}\right)\right) = 0$

Exercise 4.8 $\quad \vec{D} = \epsilon\vec{E} = \frac{\epsilon k V_o}{M(m+k\rho)}\vec{a}_\rho \qquad \rho_s\Big|_{\rho=a} = \frac{\epsilon k V_o}{M(m+ka)}$

$$Q_a = 2\pi a L\,\rho_s\Big|_{\rho=a} = \frac{2\pi a L\,\epsilon k V_o}{M(m+ka)},\ \text{similarly}\ Q_b\Big|_{\rho=b} = -\frac{2\pi b L\,\epsilon k V_o}{M(m+kb)}$$

$$\rho_v = \nabla\cdot\vec{D} = \frac{\epsilon k V_o}{M}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\rho}{m+k\rho}\right)\right] = \frac{\epsilon k V_o}{M\rho}\frac{m}{(m+k\rho)^2}$$

$$Q_V = \frac{\epsilon k V_o m}{M}\int_a^b \frac{1}{\rho(m+k\rho)^2}\rho\, d\rho\int_0^{2\pi} d\phi\int_0^L dz = \frac{\epsilon k V_o m}{M}\cdot\frac{2\pi L(b-a)}{(m+ka)(m+kb)}$$

$$Q_T = Q_a + Q_b + Q_V = 0$$

Exercise 4.9

$\nabla\cdot\vec{J} = 0 \Rightarrow \nabla\cdot\sigma\vec{E} = 0$

$\sigma\nabla\cdot\vec{E} + \vec{E}\cdot\nabla\sigma = 0 \quad (1)$

$\sigma = k + \frac{m}{\rho}, \quad \nabla\sigma = -\frac{m}{\rho^2}\vec{a}_\rho$

From (1) $\quad -\frac{m}{\rho^2}E_\rho + \left(\frac{m}{\rho}+k\right)\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho) = 0$

$$\frac{d}{d\rho}(\rho E_\rho) = \frac{m E_\rho}{\rho\left(\frac{m}{\rho}+k\right)} = \frac{m E_\rho}{(m+k\rho)}$$

$$\rho\frac{dE_\rho}{d\rho} + E_\rho = \frac{m E_\rho}{m+k\rho} \Rightarrow$$

$$\frac{dE_\rho}{d\rho} = -\frac{k}{m+k\rho}E_\rho$$

or $\quad \frac{dE_\rho}{E_\rho} = -\frac{k\,d\rho}{m+k\rho} \Rightarrow$

$$\ln(E_\rho) = -\ln(m+k\rho) + \ln c_1$$

or $\quad E_\rho = \frac{c_1}{m+k\rho}$

Since $\vec{E} = -\nabla V \Rightarrow \frac{dV}{d\rho} = -\frac{c_1}{m+k\rho}$

$$V = -\frac{c_1}{k}\ln(m+k\rho) + c_2$$

at $\rho = b$, $V = 0 \Rightarrow c_2 = \frac{c_1}{k}\ln(m+kb)$

at $\rho = a$, $V = V_o \Rightarrow c_1 = \frac{V_o k}{M}$

where $M = \ln\left[\frac{m+kb}{m+ka}\right]$

Hence, $V = \frac{V_o}{M}\ln\left(\frac{m+kb}{m+k\rho}\right)$

and $E_\rho = \frac{V_o k}{M(m+k\rho)}$, $D_\rho = \frac{\epsilon V_o k}{M(m+k\rho)}$

$$J_\rho = \sigma E_\rho = \frac{\sigma V_o k}{M(m+k\rho)} = \frac{V_o k}{M\rho}$$

$$I = \int \vec{J}\cdot d\vec{s} = \frac{2\pi V_o L k}{M},\quad R = \frac{V_o}{I} = \frac{M}{2\pi L k}$$

$$\rho_{sa} = \frac{\epsilon V_o k}{M(m+ka)}$$

$$Q_a = 2\pi a L\rho_{sa} = \frac{2\pi a L\,\epsilon V_o k}{M(m+ka)}$$

$$C = \frac{Q_a}{V_o} = \frac{2\pi a\epsilon k L}{M(m+ka)} \Rightarrow C = \frac{2\pi\epsilon L}{\ln(b/a)}\ \text{when}\ m=0$$

Exercise 4.10 $\nabla^2 V = 0 \Rightarrow \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right)\right] = 0$ $\sigma = 0.4$ S/m

Thus, $\sin\theta \frac{\partial V}{\partial\theta} = c_1 \Rightarrow \frac{\partial V}{\partial\theta} = \frac{c_1}{\sin\theta} \Rightarrow V = c_1 \ln(\tan\frac{\theta}{2}) + c_2$

at $\theta = 45^\circ$, $V = 0 \Rightarrow c_2 = 0.881\, c_1$

at $\theta = 30^\circ$, $V = 100 \Rightarrow c_1 = -229.28$ Hence: $V = -229.28[\ln(\tan\frac{\theta}{2}) + 0.881]$

$\vec{E} = -\nabla V \Rightarrow E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{229.28}{r\sin\theta}$, $J_\theta = \sigma E_\theta = \frac{91.712}{r\sin\theta}$

$I = \int_s \vec{J}\cdot d\vec{s} = 91.712 \iint \frac{1}{r\sin\theta} \cdot r\sin\theta\, d\theta\, dr = 91.712 \int_{0.1}^{0.2} dr \int_{\pi/6}^{\pi/3} d\phi = 4.8\,A$

$$R = \frac{100}{4.8} = 20.83\,\Omega$$

Exercise 4.11 $D_\theta = \epsilon E_\theta = 5\epsilon_0 E_\theta = \frac{10.14}{r\sin\theta}$ nC/m^2

$\rho_{sa}\Big|_{\theta=30^\circ} = \frac{10.14\times10^{-9}}{r\sin 30^\circ} = \frac{20.28\times10^{-9}}{r} \Rightarrow Q_{sa} = 20.28\times10^{-9}\int_{0.1}^{0.2}\frac{1}{r}\, r\sin 30^\circ dr \int_{\pi/6}^{\pi/3} d\phi = 530.93$ PC

Similarly, $\rho_{sb}\Big|_{\theta=45^\circ} = -530.93$ PC, $\rho_v = \nabla\cdot\vec{D} = 0 \Rightarrow Q_v = 0$

Exercise 4.12 $\vec{J} = \sin(10x)\,\vec{a}_x + y\,\vec{a}_y + e^{-3z}\vec{a}_z$ A/m^2

$\frac{\partial\rho_v}{\partial t} = -\nabla\cdot\vec{J} = -[10\cos(10x) + 1 - 3e^{-3z}]$ A/m^3

Exercise 4.13 $\vec{J} = e^{-\beta\rho}\cos\phi\,\vec{a}_\rho + \ln(\cos\beta z)\vec{a}_z$ A/m^2

$\frac{\partial\rho_v}{\partial t} = -\nabla\cdot\vec{J} = -[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho e^{-\beta\rho}\cos\phi) + \frac{\partial}{\partial z}[\ln(\cos\beta z)]] = (\beta - \frac{1}{\rho})\cos\phi\, e^{-\beta\rho} + \beta\tan\beta z$

Exercise 4.14

$$\rho_v = \begin{cases} \rho_0 e^{-t/\tau} & \rho \le 2\,\text{cm} \\ 0 & \rho \ge 2\,\text{cm} \end{cases} \qquad \rho_0 = 10\,\mu\text{C/m}^3$$

$$E_\rho = \begin{cases} \frac{\rho_0}{2\epsilon}\rho\, e^{-t/\tau} & \rho \le 2\,\text{cm} \\ \frac{\rho_0 (0.02)^2}{2\epsilon\,\rho} e^{-t/\tau} & 2^{+} \le \rho \le 10^{-}\text{cm} \\ \frac{\rho_0\,(0.02)^2}{2\epsilon_0\,\rho} & \rho \ge 10^{+}\text{cm} \end{cases}$$

$\vec{J} = \sigma\vec{E} \Rightarrow$

$$J_\rho = \begin{cases} \frac{\sigma\rho_0}{2\epsilon}\rho\, e^{-t/\tau} & \rho \le 2^{-}\text{cm} \\ \frac{\sigma\rho_0 (0.02)^2}{2\epsilon\rho} e^{-t/\tau} & 2^{+} \le \rho \le 10^{-}\text{cm} \\ 0 & \rho \ge 10^{+}\text{cm} \end{cases}$$

$\tau = \frac{\epsilon}{\sigma}$ and $T = 5\tau$

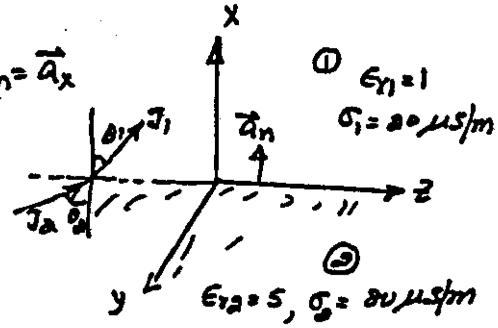
$T_{Cu} = \frac{5\times10^{-9}}{36\pi}\times1.72\times10^{-8} = 7.6\times10^{-14}$ s

$T_{Al} = \frac{5\times10^{-9}}{36\pi}\times2.83\times10^{-8} = 12.51\times10^{-19}$ s

$\rho_{outer} = \epsilon_0 E_\rho\Big|_{\rho=+10^{-}\text{cm}} - \epsilon E_\rho\Big|_{\rho=10^{-}\text{cm}} = 20(1 - e^{-t/\tau})$ nC/m^2

$T_c = 1.55\times10^{-15}$ s, $T_{Quartz} = 33.16\times10^{6}$ s

Exercise 4.20 $\vec{J}_1 = 100\vec{a}_x + 20\vec{a}_y - 50\vec{a}_z \ A/m^2$



$J_{n1} = J_{n2} = J_n \Rightarrow J_{x2} = 100$

$J_{t2} = \frac{\sigma_2}{\sigma_1} J_{t1} \Rightarrow J_{y2} = \frac{80}{20} \times 20 = 80$

$J_{z2} = \frac{80}{20}(-50) = -200$

$\vec{J}_2 = 100\vec{a}_x + 80\vec{a}_y - 200\vec{a}_z \ A/m^2$

$\rho_s\big|_{\text{Interface}} = J_n\left[\frac{\epsilon_1}{\sigma_1} - \frac{\epsilon_2}{\sigma_2}\right] = 100\left[\frac{10^6}{20} - \frac{5\times10^6}{80}\right]\frac{10^{-9}}{36\pi}$

$= 11.05\ \mu c/m^2$

$J_1 = 113.578\ A/m^2,\ J_2 = 237.487\ A/m^2 \quad \theta_1 = \cos^{-1}\left(\frac{100}{113.578}\right) = 28.3^\circ,\ \theta_2 = \cos^{-1}\left(\frac{100}{237.487}\right) = 65.1^\circ$

Exercise 4.21 $\nabla^2 V = 0 \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0 \Rightarrow V = -\frac{c_1}{r} + c_2$

at $r = b$ $V = 0 \Rightarrow c_2 = \frac{c_1}{b}$, At $r = a$, $V = V_0 \Rightarrow c_1 = \frac{V_0 ab}{a-b}$

$V = -\frac{V_0 ab}{(b-a)r} + \frac{V_0 a}{b-a}$, $E = -\nabla V \Rightarrow E_r = \frac{V_0 ab}{(b-a)r^2}$, $D_r = \frac{V_0 ab\epsilon}{(b-a)r^2}$

From Analogy: $J_r = \frac{V_0 ab\sigma}{(b-a)r^2}$, $I = \int_s \vec{J}\cdot\vec{ds} = \frac{V_0 ab\sigma}{b-a}\int_0^{\pi}\frac{1}{r^2} r^2 \sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{4\pi V_0 ab\sigma}{b-a}$

Exercise 4.22 $V_0 = 1000\,V,\ a = 2\,cm,\ b = 5\,cm,\ \epsilon_r = 1,\ \sigma = 4\times10^{-6}\,s/m$

Thus, $I = \frac{4\pi \times 1000 \times 2\times10^{-2} \times 5\times10^{-2} \times 4\times10^{-6}}{3\times10^{-2}} = 1.676\,mA$

Exercise 4.23

$\nabla^2 V = 0 \Rightarrow \rho\frac{\partial V}{\partial \rho} = c_1 \Rightarrow V = c_1 \ln\rho + c_2$

at $\rho = b$ $V = 0 \Rightarrow c_2 = -c_1 \ln b$

at $\rho = a$ $V = V_0 \Rightarrow c_1 = \frac{V_0}{\ln(a/b)}$

$V = \frac{V_0}{\ln(a/b)}\ln(\rho/b)$

$\vec{E} = -\nabla V \Rightarrow E_\rho = -\frac{\partial V}{\partial \rho} = \frac{V_0}{\rho \ln(b|a)}$

$D_\rho = \frac{V_0 \epsilon}{\ln(b|a)}\cdot\frac{1}{\rho} \Rightarrow J_\rho = \frac{\sigma V_0}{\rho \ln(b|a)}$

$I = \int_s \vec{J}\cdot\vec{ds} = \frac{\sigma V_0}{\ln(b|a)}\int_0^{2\pi}\frac{1}{\rho}\rho\, d\phi \int_0^L dz$

or

$I = \frac{2\pi\sigma L V_0}{\ln(b|a)}$

substitute $L = 100\,m$, $a = 2\,cm$

$b = 5\,cm$, $\epsilon_r = 2$, $\sigma = 10\times10^{-6}\,s/m$

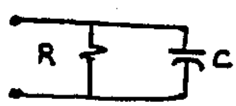
and $V_0 = 5000\,V$, we have

$I = 34.286\,A$

$P_d = VI = 171.43\,W$

$R = \frac{V_0}{I} = \frac{\ln(b|a)}{2\pi\sigma L} = 145.83\,\Omega$

$G = \frac{2\pi L\sigma}{\ln(b/a)} \Rightarrow C = \frac{2\pi\epsilon L}{\ln(b/a)} \cong 12\,nF$



Exercise 4.24 $R = \frac{\rho L}{A} = \frac{1.72\times10^{-8}\times10\times10^{3}}{\pi\times0.65^{2}\times10^{-6}} = 129.584\,\Omega$, $I = \frac{24}{R} = 0.185$ A

$J = \frac{I}{A} = \frac{0.185}{\pi\times0.65^{2}\times10^{-6}} = 139.53$ kA/m^2 $\quad P_d = I^2R = 4.44$ W

Exercise 4.25 $P_d = 4.5 = \frac{V^2}{R} \Rightarrow R = 128\,\Omega$ if V is the same, i.e. $V = 24$V

$R = \frac{\rho l}{A} \Rightarrow A = \frac{100\times10^{-8}\times10\times10^{3}}{128} = 78.125$ mm^2 $\Rightarrow d = 10$ mm (dia)

Exercise 4.26 If V is the total voltage drop across n resistors in series and IR_i is the voltage drop across ith resistor, then

$V = \sum_{i=1}^{n} IR_i = I\sum_{i=1}^{n} R_i$. If R is the eqvl. resistance, then $V = IR$. Thus, $R = \sum R_i$

Exercise 4.27 If V is the voltage drop across n resistors in parallel and I_i is the current thro' ith resistor, then the total current is

$I = \sum_{i=1}^{n} I_i = \sum_{i=1}^{n} \frac{V}{R_i} = V\sum_{i=1}^{n}\frac{1}{R_i}$ ①

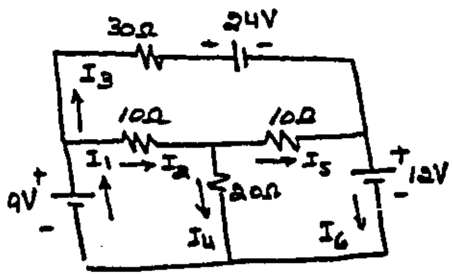
If R is the equivalent resistance, $I = \frac{V}{R}$ ②

From ① and ②

$\frac{1}{R} = \sum_{i=1}^{n}\frac{1}{R_i}$ or

$G = \sum_{i=1}^{n} G_i$

Exercise 4.28



$10I_2 + 20I_4 = 9$

$10I_5 - 20I_4 = -12$ — ③

$I_2 + I_5 = -0.3$ ①

$30I_3 - 10I_5 - 10I_2 = -24$

$30I_3 - 10(I_2 + I_5) = -24$ ②

From ① and ② $\quad I_3 = -0.9$ A

$I_4 = I_2 - I_5$ and from ③

$30I_5 - 20I_2 = -12$

$20I_5 + 20I_2 = -6$ from ①

$50I_5 = -18 \Rightarrow I_5 = -0.36$A

From ① $\quad I_2 = -0.3 + 0.36 = 0.06$A

From ③ $\quad I_4 = 0.06 + 0.36 = 0.42$A

$I_1 = I_2 + I_3 = -0.84$ A

$I_6 = I_5 + I_3 = -1.26$A

$P_{supplied} = 9I_1 - 24I_3 - 12I_6$
$= 29.16$ W

$P_{diss} = 30I_3^2 + 10I_2^2 + 10I_5^2 + 20I_4^2$
$= 29.16$W

Thus, $P_{supplied} = P_{diss}$.

Problem 4.1 $I = 1500\ A$, $A = 10\ cm^2$, $J = I/A = 1.5\times10^6\ A/m^2$

$$J = \sigma E \Rightarrow E = \rho J = 1.7\times10^{-8}\times1.5\times10^{6} = 25.5\ mV/m$$

$$u_e = \frac{\sigma}{Ne} = \frac{1}{\rho N e} = \frac{1}{1.72\times10^{-8}\times8.5\times10^{28}\times1.6\times10^{-19}} = 4.275\ mm/s$$

$$\mathcal{U}_e = u_e E = 109\ \mu m/s$$

Problem 4.2 $L = 10\ m$, $V = 100\ V$, $r = 2\ mm \Rightarrow E = \frac{V}{L} = 10\ V/m$, $A = \pi r^2 = 4\pi\ mm^2$

$$J = \sigma E = \frac{E}{\rho} = \frac{10}{7.8\times10^{-8}} = 128.205\times10^6\ A/m^2 \qquad I = JA \cong 1611\ A$$

Problem 4.3 $\mathcal{U} = 3\times10^5\ m/s$ $\quad J = 10\ A/cm^2 = 10^5\ A/m^2$

$$J = Ne\mathcal{U} \Rightarrow N = \frac{10^5}{1.6\times10^{-19}\times3\times10^5} = 2.08\times10^{18}\ \text{electrons}$$

Problem 4.4 $J = 0.2\times10^{-9}\ A/m^2$ $\quad \vec{J} = \rho_+\vec{\mathcal{U}}_+ + \rho_-\vec{\mathcal{U}}_-$ $\quad |\rho_+| = |\rho_-|,\ |\vec{\mathcal{U}}_+| = |\mathcal{U}_-|$

$$J = 2\rho\mathcal{U} \Rightarrow \mathcal{U} = \frac{0.2\times10^{-9}}{2\times25\times1.6\times10^{-19}} = 2.5\times10^{7}\ m/s$$

Problem 4.5 If A is the area of each plate, then $\vec{J} = -\frac{100}{A}\vec{a}_z$

$$\vec{E} = \frac{\vec{J}}{\sigma} = \rho\vec{J} = (2.6\times10^3)\left(-\frac{100}{A}\right)\vec{a}_z = -\frac{260\times10^3}{A}\vec{a}_z\ V/m$$

$$\vec{D} = \epsilon_r\epsilon_0\vec{E} = -\frac{2.3\times10^{-6}}{A}\vec{a}_z\ C/m^2,\quad \rho_{s+}\big|_{top} = \frac{2.3\times10^{-6}}{A}\ C/m^2,\quad Q_{Top} = 2.3\times10^{-6}\ C$$

$$\rho_{s-}\big|_{BOT} = -\frac{2.3}{A}\ \mu C/m^2 \text{ and } Q_- = -2.3\ \mu C$$

Problem 4.6 $L = 30\ km$, $r = 1.29\ mm$

$$R = \frac{L}{\sigma A} = \frac{\rho L}{A} \qquad R_{cu} = \frac{1.7\times10^{-8}\times30\times10^{3}}{\pi\times1.29^2\times10^{-6}} = 97.55\ \Omega$$

$$R_{al} = \frac{2.83\times10^{-8}\times30\times10^{3}}{\pi\times1.29^2\times10^{-6}} = 162.4\ \Omega$$

$$R_{Ni} = \frac{100\times10^{-8}\times30{,}000}{\pi\times1.29^2\times10^{-6}} = 5738.4\ \Omega$$

Problem 4.7

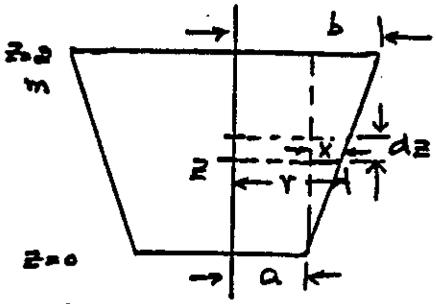
$$dR = \frac{dl}{\sigma A} = \frac{r\,d\theta}{\sigma A},\ \text{function of } r \text{ and } \theta \Rightarrow R = \int_0^{\pi/2}\frac{d\theta}{\int_a^b \frac{\sigma t}{r}dr} = \int_0^{\pi/2}\frac{d\theta}{\sigma t\ln(b/a)}$$

$dA = t\,dr$ function of r

$$= \frac{\pi}{2\sigma t\ln(b/a)}$$

Thus, group r components together

Problem 4.8 $\frac{x}{z} = \frac{b-a}{2} = \frac{2\times10^{-2}}{2} = 0.01$



Thus, $r = 0.1 + 0.01z$, $A = \pi r^2$

$$R = \int_0^2 \frac{dz}{\sigma\pi(0.1+0.01z)^2} = \frac{10^4}{\sigma\pi}\int_0^2 \frac{dz}{(z+10)^2}$$

$$= \frac{10^4}{\sigma\pi}\left[\frac{1}{z+10}\right]_2^0 = \frac{10^4}{\sigma\pi}\left[\frac{1}{10} - \frac{1}{12}\right] \qquad \sigma = \frac{1}{\rho} = \frac{1}{8.9\times10^{-8}}$$

$$= 4.72\ \mu\Omega$$

Problem 4.9 $R_{cu} = \frac{10\times10^{3}\times1.72\times10^{-8}}{\pi\, 2^2\times10^{-4}} = 0.137\,\Omega$, $R_{cons} = \frac{10\times10^{3}\times49\times10^{-8}}{\pi(3^2-2^2)10^{-4}} = 3.119\,\Omega$

Thus, $R = \frac{R_{cu}R_{cons}}{R_{cu}+R_{cons}} = 0.1311\,\Omega$, Applied voltage : $V = 100R = 13.11\ V$

$I_{cu} = \frac{13.11}{R_{cu}} = 95.69\,A$, $I_{cons} = \frac{13.11}{3.119} = 4.2\ A$, $J_{cu} = \frac{I_{cu}}{A_{cu}} = \frac{95.69}{\pi\times4\times10^{-4}} = 76.15\ \frac{kA}{m^2}$

$J_{const} = 2.67\ kA/m^2$, $E_{cu} = \rho_{cu}J_{cu} = 1.72\times10^{-8}\times76.15\times10^{3} = 1.31\ mV/m$

$E_{cons} = 49\times10^{-8}\times 2.67\times10^{3} = 1.31\ mV/m$ (as expected)

Problem 4.10 $\rho_v = 8.5\times10^{28}\times1.6\times10^{-19} = 1.36\times10^{10}\ c/m^3$

$J = \frac{200}{10^{-4}} = 2\times10^6\ A/m^2$, $\vec{J} = \rho_v\vec{u} \Rightarrow u = \frac{2\times10^6}{1.36\times10^{10}} = 147.06\times10^{-6}\ m/s$

$J = \sigma E \Rightarrow E = \rho J = 1.72\times10^{-8}\times 2\times10^{6} = 34.4\ mV/m$ Also $R = \frac{\ell}{\sigma A} = \frac{1.72\times10^{-8}\times100\times10^{3}}{1\times10^{-4}} = 17.2\,\Omega$

$V = -\int\vec{E}\cdot\vec{d\ell} = 34.4\times10^{-3}\times100\times10^{3} = 3440V \Rightarrow R = \frac{V}{I} = 17.2\Omega$

Problem 4.11 $I = \frac{2\times10^{-3}}{4.72\times10^{-6}} = 423.73\ A$, $\vec{J} = -\frac{423.73}{\pi(z+10)^2 10^{-4}}\vec{a}_z$

$$\vec{E} = \rho\vec{J} = -\frac{8.9\times10^{-8}\times423.73}{\pi\times10^{-4}(z+10)^2}\vec{a}_z = -\frac{0.12}{(z+10)^2}\vec{a}_z \quad V/m.$$

verify:

$$V = -\int_c \vec{E}\cdot\vec{dl} = 0.12\int_0^2\frac{dz}{(z+10)^2} = 0.12\left[\frac{-1}{z+10}\right]_0^2$$

$$= 0.12\left[\frac{1}{10} - \frac{1}{12}\right] = 0.002$$

or 2mV

Problem 4.12

$\nabla\cdot\vec{J}=0 \Rightarrow \nabla\cdot\sigma\vec{E}=0 \Rightarrow \vec{E}\cdot\nabla\sigma+\sigma\nabla\cdot\vec{E}=0$

$\sigma=\frac{m}{r}+k \qquad \nabla\sigma=-\frac{m}{r^2}$

Thus, $-\frac{m}{r^2}E_r+\left(\frac{m+kr}{r}\right)\left[\frac{dE_r}{dr}+\frac{2}{r}E_r\right]=0$

$\frac{dE_r}{dr}=-\frac{m+2kr}{r(m+kr)}E_r \Rightarrow$

$E_r=\frac{c_1}{r(m+kr)}$, Since $\vec{E}=-\nabla V$

$\frac{dV}{dr}=-\frac{c_1}{r(m+kr)}=-\frac{c_1}{m}\left[\frac{1}{r}-\frac{k}{m+kr}\right]$

or

$V=-\frac{c_1}{m}\ln(r)+\frac{c_1}{m}\ln(m+kr)+c_2$

at $r=b$, $V=0 \Rightarrow c_2=\frac{c_1}{m}\ln\left(\frac{b}{m+kb}\right)$

at $r=a$, $V=V_0 \Rightarrow c_1=\frac{V_0 m}{M}$

where $M=\ln\left[\frac{(m+ka)b}{(m+kb)a}\right]$

Thus, $V=\frac{V_0}{M}\ln\left[\frac{(m+kr)b}{(m+kb)r}\right]$

and $E_r=\frac{V_0 m}{M}\cdot\frac{1}{r(m+kr)}$

$J_r=\sigma E_r=\frac{V_0 m}{M r^2}$

$I=\int\vec{J}\cdot d\vec{s}=\frac{4\pi V_0 m}{M}$

$R=\frac{V_0}{I}=\frac{M}{4\pi m}$

when $m\to 0$, $\sigma=k$

$R=\frac{1}{4\pi k}\left[\frac{1}{a}-\frac{1}{b}\right]$

$=\frac{1}{4\pi\sigma}\left[\frac{1}{a}-\frac{1}{b}\right]$

Problem 4.13 $L=2\pi\times 8\times10^{-3}\times 200=3.2\pi$, $A=\frac{\pi}{4}\times 0.45^2\times10^{-6}$

$R=\frac{3.2\pi\times1.72\times10^{-8}}{\frac{\pi}{4}\times0.45^2\times10^{-6}}=1.09\,\Omega$

Problem 4.14 $R=6\,\Omega \Rightarrow \sigma=\frac{L}{RA}=\frac{10\times10^{6}}{6\times\pi\times0.5^2}=2.12\times10^{6}$ S/m

Problem 4.15 $L=R\sigma A=\frac{RA}{\rho}=\frac{10\times0.25^2\times10^{-6}\pi}{3.5\times10^{-5}}=56.1$ mm

Problem 4.16 $\frac{L_1}{\sigma_1 A_1}=\frac{L_2}{\sigma_2 A_2} \qquad L_1=L_2 \quad \sigma_1=\sigma_2 \Rightarrow A_1=A_2$

$\pi\times2^2=\pi\left[b^2-2^2\right] \Rightarrow b=2.83$ mm.

Problem 4.17

$V=1200\times4.5=5400$ V

$E=\frac{5400}{200{,}000}=0.027$ V/m

$J=\frac{I}{A}=\frac{1200}{\pi 2^2\times10^{-4}}=954.93\times10^{3}\,\frac{A}{m^2}$

$J=\sigma E \Rightarrow \sigma=35.37\times10^{6}$ S/m

$\rho=\frac{1}{\sigma}=2.83\times10^{-8}\,\Omega\text{-m}$ (Aluminum)

Problem 4.18 $\quad q = -500\times10^{12}\times1.6\times10^{-19} = -80\,\mu C$

$\tau = \frac{\epsilon}{\sigma} = \rho\,\epsilon_0 = 2.83\times10^{-8}\times10^{-9}/36\pi = 2.5\times10^{-19}\,s$

$\rho = \rho_0\, e^{-t/\tau}$. Let $t = t_1$ when $\rho = 0.8\rho_0$, then

$0.8 = e^{-t_1/\tau} \Rightarrow t_1 = \tau \ln(1.25) = 55.786\times10^{-21}\,s$

Problem 4.19 $\quad 0.5 = e^{-100\times10^{-9}/\tau} \Rightarrow \tau = 144.27\,ns$

$\tau = \frac{\epsilon}{\sigma} \Rightarrow \sigma = \frac{2.5\times10^{-9}}{36\pi}\cdot\frac{10^{9}}{144.27} = 153.22\,\mu S/m$

$\frac{\rho}{\rho_0} = e^{-200\times10^{-9}/144.27\times10^{-9}} = 0.25$ or 25%

Problem 4.20 $\quad i = 0.2\,e^{-50t}\,A$, $\tau = \frac{1}{50} = 0.02\,s$

$Q(t) = \int_0^t i\,dt = 0.2\int_0^t e^{-50t}\,dt = 4[1 - e^{-50t}]\,mC$

as $t \to \infty$, The initial charge : $Q(\infty) = 4mC$

$Q(2\tau) = 4(1 - e^{-50\times2\times0.02}) = 3.459\,mC$

Let the current at $t = t_1 = 0.1I_0$. Then, $0.1 = e^{-50t_1} \Rightarrow t_1 = 46.05\,ms$.

Problem 4.21 $\quad \vec{J} = e^{-x}\sin\omega x\,\vec{a}_x\,A/m^2$, $\frac{\partial\rho}{\partial t} = -\nabla\cdot\vec{J} = e^{-x}[\sin\omega x - \omega\cos\omega x]$

Problem 4.22 $\quad \nabla\cdot\vec{J} + \frac{\partial\rho_v}{\partial t} = 0$, However, $\vec{J} = \rho_v\vec{u}$

$\nabla\cdot\vec{J} = \nabla\cdot(\rho_v\vec{u}) = \vec{u}\cdot\nabla\rho_v + \rho_v\nabla\cdot\vec{J}$, Thus,

$\vec{u}\cdot\nabla\rho_v + \rho_v\nabla\cdot\vec{u} + \frac{\partial\rho_v}{\partial t} = 0$

Problem 4.23 Power density: $\quad p = \vec{J}\cdot\vec{E} = 128.205\times10^{6}\times10 = 128.205\times10^{7}\ W/m^3$

$P = \int_v p\,dv = 128.205\times10^{7}\times\pi\times2^2\times10^{-6}\times10 = 161.107\,kW$

$P = \frac{V^2}{R} = VI = 1611\times100 = 161.1\,kW$

Problem 4.24 $\quad P = VI = 12\times2 = 24W$

Problem 4.25 $\quad R = 10\,\Omega$, $V = 12\,V$, $\quad P = \frac{V^2}{R} = 14.4\,W$

Problem 4.26

$\nabla^2 V = 0 \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0 \Rightarrow$

$V = C_1 \ln \rho + C_2$

at $\rho = b$, $V = 0 \Rightarrow C_2 = -C_1 \ln b$

at $\rho = a$, $V = V_0 \Rightarrow C_1 = \frac{V_0}{\ln(a/b)}$

Thus, $V = V_0 \frac{\ln(b/\rho)}{\ln(b/a)}$

$E = -\nabla V = -\frac{C_1}{\rho}\vec{a}_\rho = \frac{V_0}{\rho \ln(b/a)}\vec{a}_\rho$

$\vec{J} = \sigma \vec{E} = \frac{\sigma V_0}{\rho \ln(b/a)}\vec{a}_\rho$

$I = \int_S \vec{J} \cdot d\vec{s} = \frac{\sigma V_0}{\rho \ln(b/a)} \int_0^{2\pi} \frac{1}{\rho}\rho\, d\phi \int_0^L dz$

$= \frac{2\pi L \sigma V_0}{\ln(b/a)}$

$R = \frac{V_0}{I} = \frac{\ln(b/a)}{2\pi L \sigma}$

Substitute, $L = 100\,m$, $a = 8\,mm$, $b = 10\,mm$

$\sigma = 6.25 \times 10^{-6}\, S/m$.

$R = 56.82\,\Omega$, $P = \frac{230^2}{R} = 931\,W$

Problem 4.27

$I = \rho_s \times \text{width} \times \frac{\text{Length}}{\text{sec}} \Rightarrow \rho_s = \frac{50 \times 10^{-6}}{30 \times 10^{-2} \times 20} = 8.33\ \mu C/m^2$

Problem 4.28

$R_1 = \frac{0.5 \times 10^{-3}}{10 \times 10^3 \times 1} = 50\,n\Omega$

$R_2 = \frac{0.2 \times 10^{-3}}{500 \times 1} = 400\,n\Omega$

$R_3 = \frac{0.3 \times 10^{-3}}{0.2 \times 10^6 \times 1} = 1.5\,n\Omega$

$R = R_1 + R_2 + R_3 = 451.5\,n\Omega$

$I = 10 \times 10^{-3} / 451.5 \times 10^{-9} = 22.148\,kA$

$J_1 = J_2 = J_3 = \frac{I}{A} = 22.148\ kA/m^2$

Since $\vec{J} = \sigma \vec{E} \Rightarrow E_1 = 2.215\,V/m$

$E_2 = 44.296\,V/m$

$E_3 = 0.111\ V/m$

$P = \frac{V^2}{R} = 221.48\,W$

Problem 4.29

$u_{final} = \sqrt{\frac{2eV}{m}} = \sqrt{\frac{2 \times 1.6 \times 10^{-19} \times 10 \times 10^3}{9.11 \times 10^{-31}}} = 59.267 \times 10^6\ m/s$

$V = V_0\left(\frac{z}{d}\right)^{4/3} = 10 \times 10^3 \left(\frac{z}{0.1}\right)^{4/3} = 215.443\ z^{4/3}\ kV$

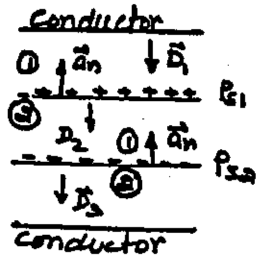
$E_z = -\frac{4}{3}\frac{V_0}{d}\left(\frac{z}{d}\right)^{1/3} = -\frac{4}{3} \times \frac{10 \times 10^3}{0.1}\left(\frac{z}{0.1}\right)^{1/3} = -287.26\ z^{1/3}\ kV/m$

$\vec{J} = -\frac{4}{9} \times \frac{10^{-9}}{36\pi} \times \left(\frac{1}{0.1}\right)^2 \sqrt{\frac{2 \times 1.6 \times 10^{-19}}{9.11 \times 10^{-31}}}\left(10 \times 10^3\right)^{3/2}\vec{a}_z = -23.29\,\vec{a}_z\ kA/m^2$

Problem 4.30

$I = 23.29 \times 10^3 \times 4 \times 4 \times 10^{-4} = 37.265\,A$

Problem 4.31



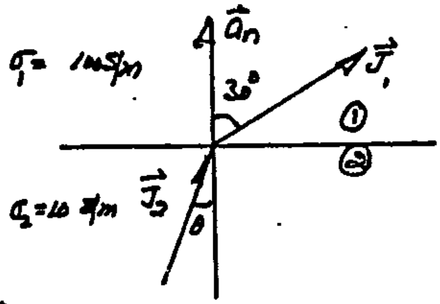
$\vec{D}_1 = -\epsilon_0 (2.215)\vec{a}_z\,,\quad \vec{D}_2 = -44.296\,\epsilon_0\,\vec{a}_z$

$\vec{D}_3 = -0.111\,\epsilon_0\,\vec{a}_z$

$\rho_{s1} = \vec{a}_z \cdot (\vec{D}_1 - \vec{D}_2) = 372.08 \text{ pC/m}^2$

$\rho_{s2} = \vec{a}_z \cdot (\vec{D}_2 - \vec{D}_3) = -390.68 \text{ pC/m}^2$

Problem 4.32



$J_{n1} = 50\cos 30^\circ = 43.3 \text{ A/m}^2$

$J_{n2} = J_{n1} = 43.3 \text{ A/m}^2$

$J_{t1} = 50 \sin 30^\circ = 25$

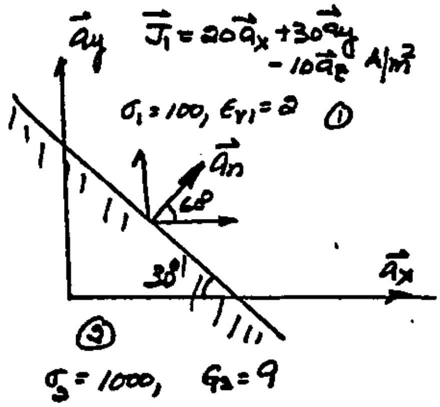
$J_{t2} = \dfrac{\sigma_2}{\sigma_1} J_{t1} = \dfrac{10}{100} \times 25 = 2.5$

$J_2 = \sqrt{43.3^2 + 2.5^2} = 43.37\,,\quad \theta = \cos^{-1}\left(\dfrac{43.3}{43.37}\right) = 3.3^\circ$

$D_{n1} = \epsilon_1 E_{n1} = \dfrac{\epsilon_1}{\sigma_1} J_{n1}\,,\quad D_{n2} = \dfrac{\epsilon_2}{\sigma_2} J_{n2} \Rightarrow \rho_s = D_{n1} - D_{n2} = J_{n1}\left[\dfrac{\epsilon_1}{\sigma_1} - \dfrac{\epsilon_2}{\sigma_2}\right]$

Hence $\rho_s = 43.3\left[\dfrac{9.6}{100} - \dfrac{4}{10}\right]\dfrac{10^{-9}}{36\pi} = -116.4 \text{ pC/m}^2$

Problem 4.33



$\vec{a}_n = \cos 60^\circ \vec{a}_x + \sin 60^\circ \vec{a}_y$

$= 0.5\vec{a}_x + 0.866\,\vec{a}_y$

$\vec{a}_n \cdot (\vec{J}_1 - \vec{J}_2) = 0 \Rightarrow 0.5 J_{x2} + 0.866\, J_{y2} = 35.98 \quad (1)$

$\vec{a}_n \times \left(\dfrac{\vec{J}_1}{\sigma_1} - \dfrac{\vec{J}_2}{\sigma_2}\right) = 0 \Rightarrow 0.5\, J_{y2} - 0.866\, J_{x2} = -23.2 \quad (2)$

from (1) and (2) $J_{x2} = 38.08,\ J_{y2} = 19.56$

$J_{z2} = J_{z1}\dfrac{\sigma_2}{\sigma_1} = -100$; Thus, $\vec{J}_2 = 38.08\,\vec{a}_x + 19.56\,\vec{a}_y - 100\,\vec{a}_z \text{ A/m}^2$

Problem 4.34

$C_1 = \dfrac{2\pi\epsilon_1}{\ln(c/a)} = \dfrac{2\pi \times 10^{-9} \times 2}{36\pi \ln(2)} = 160.3 \text{ pF/m}$

$C_2 = \dfrac{2\pi\epsilon_2}{\ln(b/c)} = \dfrac{2\pi \times 4 \times 10^{-9}}{36\pi \ln(2)} = 320.6 \text{ pF/m}$

$C = \dfrac{C_1 C_2}{C_1 + C_2} = 106.87 \text{ pF/m}$

$G_1 = \dfrac{\sigma_1}{\epsilon_1} C_1 = 453.24 \times 10^{-6} \text{ S·m}$

$G_2 = \dfrac{\sigma_2}{\epsilon_2} C_2 = 906.48\ \mu\text{S·m}$

$R_1 = \dfrac{1}{G_1} = 2206\ \Omega/\text{m},\ R_2 = 1103\ \Omega/\text{m}$

$R = R_1 + R_2 = 3309\ \Omega/\text{m}$

Problem 4.35 $L=100$ m $\Rightarrow$ $R\simeq 33\Omega$ $I= 10/33 = 303$ mA $\sigma_1 = 50 \mu S/m$

At any radius ρ, $\vec{J}= \frac{I\,\vec{a}_\rho}{2\pi\rho L} = \frac{482.24}{\rho}\vec{a}_\rho$ $\mu A/m^2$ $\epsilon_1 = 2\epsilon_0$

$\sigma_2 = 100 \mu S/m$

Region-1: $\vec{E}_1 = \frac{\vec{J}}{\sigma_1} = \frac{9.64}{\rho}\vec{a}_\rho$ V/m, $\vec{D}_1 = \epsilon_1\vec{E}_1 = \frac{170.56}{\rho}\vec{a}_\rho$ pC/m^2 $\epsilon_2 = 4\epsilon_0$

Region-2: $\vec{E}_2 = \frac{\vec{J}}{\sigma_2} = \frac{4.82}{\rho}\vec{a}_\rho$ V/m, $\vec{D}_2 = \epsilon_2\vec{E}_2 = \frac{170.56}{\rho}\vec{a}_\rho$ pC/m^2

at $\rho = 20$cm, $\rho_s = D_{n1} - D_{n2} = 0$

Problem 4.36 Per unit length. $\tau = RC = 3309 \times 106.87\times 10^{-12} = 353.63$ ns

$\frac{1}{2}V_0 = V_0 e^{-T/\tau} \Rightarrow T = \tau \ln(2) = 245.12$ ns

Problem 4.37 $C_1 = \frac{4\pi\epsilon_1 ac}{c-a} = 4\pi \times \frac{2\times 10^{-9}}{36\pi}\frac{3\times 6\times 10^{-4}}{3\times 10^{-2}} = 20$ PF

$C_2 = \frac{4\pi\epsilon_2 bc}{b-c} = 4\pi \times \frac{4\times 10^{-9}}{36\pi}\times \frac{6\times 9\times 10^{-4}}{3\times 10^{-2}} = 80$ PF, $C = \frac{C_1 C_2}{C_1+C_2} = 16$ PF

$G_1 = \frac{\sigma_1}{\epsilon_1}C_1 = 20\times 10^{-12}\times 50\times 10^{-6}\times 36\pi\times 10^{9}/2 = 37.7\times 10^{-6} \Rightarrow R_1 = 26.526$ kΩ

$G_2 = \frac{\sigma_2}{\epsilon_2}C_2 = 80\times 10^{-12}\times 100\times 10^{-6}\times 36\pi\times 10^{9}/4 = 226.19\times 10^{-6} \Rightarrow R_2 = 4.421$ kΩ

$R = R_1 + R_2 = 30.947$ kΩ

Problem 4.38 $I = \frac{50}{30.947}\times 10^{-3} = 1.616$ mA, $\vec{J} = \frac{I}{A}\vec{a}_r = \vec{a}_r\frac{1.616\times 10^{-3}}{4\pi r^2} = \frac{128.57}{r^2}\vec{a}_r$ $\mu A/m^2$

$\vec{E}_1 = \frac{\vec{J}_1}{\sigma_1} \Rightarrow \vec{D}_1 = \frac{\epsilon_1\vec{J}_1}{\sigma_1}$, Also, $\vec{D}_2 = \frac{\epsilon_2\vec{J}_2}{\sigma_2}$ at $r = 6$ cm, $\vec{J}_1 = \vec{J}_2 = \vec{J} \Rightarrow$

$$\rho_s = \vec{a}_r\cdot(\vec{D}_1 - \vec{D}_2) = \frac{128.57\times 10^{-6}}{(6\times 10^{-2})^2}\left[\frac{3}{50\times 10^{-6}} - \frac{4}{100\times 10^{-6}}\right]\frac{10^{-9}}{36\pi} = 6.32 \text{ nC/m}^2$$

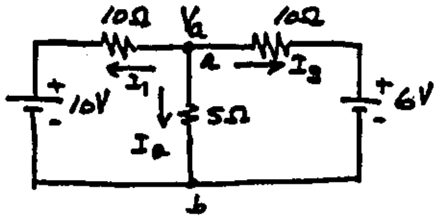
Problem 4.39 $\tau = RC = 30.947\times 10^{3}\times 16\times 10^{-12} = 4.95\times 10^{-13}$ s

$T = RC\ln(2) = 3.43\times 10^{-13}$ s, $5\tau = 2.48$ ps

Problem 4.40

$$R = 100 + \frac{5(10+5)}{5+10+5} = 103.75\,\Omega$$

Problem 4.41 Node b as reference.

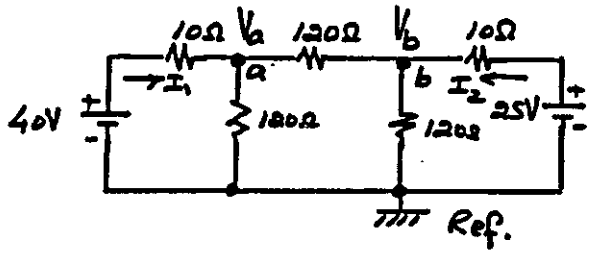


At node a: $I_1 + I_2 + I_3 = 0$

$$\frac{V_a - 10}{10} + \frac{V_a}{5} + \frac{V_a - 6}{10} = 0 \Rightarrow$$

$$V_a = 4V$$

Problem 4.42



$$\frac{V_a - 40}{10} + \frac{V_a}{180} + \frac{V_a - V_b}{120} = 0$$

$$\frac{V_b - V_a}{120} + \frac{V_b}{120} + \frac{V_b - 25}{10} = 0$$

or
$$\left.\begin{aligned} 14V_a - V_b &= 480 \\ -V_a + 14V_b &= 300 \end{aligned}\right\} \Rightarrow \begin{aligned} V_a &= 36V \\ V_b &= 24V \end{aligned}$$

$I_1 = \frac{40-36}{10} = 0.4A$, $I_2 = \frac{25-24}{10} = 0.1 A$

$P_{40V} = 40 \times 0.4 = 16 W$, $P_{25V} = 25 \times 0.1 = 2.5W$. $P_{Supplied} = 18.5W$